Generating Domination Dot Stable Graphs using Graph Operations

M. Yamuna and K. Karthika

***Abstract*— A graph G is said to be domination dot stable (DDS) if γ (G uv) = γ (G), for all u, v ∈ V (G), u adjacent to v. In this paper we show that every graph is an induced subgraph of a DDS graph and we discuss few methods of generating new DDS graphs from existing DDS graphs using various graph operations.**

***Index Terms* — domination, dot stable graphs.**

I. INTRODUCTION

We consider only simple connected undirected graphs G = (V, E). The subgraph of G induced by the vertices in D is denoted by < D >. P_n denote a path of length n. The open neighborhood of vertex v ∈ V (G) is defined by N (v) = { u ∈ V (G) | (u v) ∈ E (G) } while its closed neighborhood is the set N [v] = N (v) ∪ { v }. The private neighborhood of v ∈ D is defined by pn [v, D] = N (v) – N (D – { v }). We indicate that u is adjacent to v by writing u ⊥ v.

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ (G). A γ - set denotes a dominating set for G with minimum cardinality.

A vertex v is said to be a, down vertex if γ (G – u) < γ (G), level vertex if γ (G – u) = γ (G), up vertex if γ (G – u) > γ (G). A vertex v is said to be selfish in the γ- set D, if v is needed only to dominate itself. A vertex v is said to be good if there is a γ – set of G containing v. If there is no γ – set of G containing v, then v is said to be a bad vertex. A graph G is said to be excellent if every vertex of G is good otherwise it is called non – excellent graph.

II. MATERIAL AND METHODS

An elementary edge contraction of a graph G is obtained by identifying 2 adjacent points u and v, that is, by removal of u and v and the addition of a new point w adjacent to those points which u or v was adjacent. A graph G is contractible to a graph H if H can be obtained from G by a sequence of edge contractions.

In [1] Tamara Burton and David P. Sumner define the following

For a pair of vertices v, u of G, we denote by G vu the graph obtained by identifying v and u. We let (vu) denote the identified vertex. So G vu may be viewed as the graph obtained from G by deleting the vertices v and u and appending a new vertex, denoted by (vu), that is adjacent to all the vertices of G – v – u that were originally adjacent to either of v or u. In the case that v is adjacent to u, G vu is the graph obtained by contracting vu. In [4], M. Yamuna and K. Karthika have introduced the concept of domination dot stable graphs

Throughout this paper circled vertices represent a γ - set and

- ○ - Represent bad vertex
- ● - Represent good vertex

Domination Dot Stable Graph

A graph G is said to be dominating dot stable (DDS) if γ (G uv) = γ (G) for all u, v ∈ V (G), u ⊥ v.

Example for DDS Graph

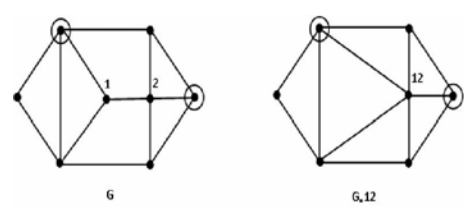


Fig. 1.The graph in the above figure is DDS. Here γ (G) = γ (G 12) = 2. Similarly γ (G) = γ (G uv) for all u, v ∈ V (G), u ⊥ v.

In [4], they have proved the following results.

R1. A graph G is DDS if and only if every γ - set of G is an independent dominating set.

R2. If G is a DDS graph, then for all x, y ∈ V (G) either D or D′ is a γ - set for G xy, where D = { u_1, u_2, …, u_k },

M. Yamuna is with the Vellore Institute of, Vellore – 632 014, India. (e-mail: myamuna@ vit.ac.in).
K. karthika is with the Vellore Institute of, Vellore – 632 014, India. (e-mail: karthika.k@ vit.ac.in)

$D' = D - \{ u_i \} \cup \{ u_i y \}$ and $u_i = x$. This is true for all $x, y \in V(G)$, $x \perp y$.

R3. Let G be a DDS, non – excellent graph and u be a bad vertex in G. In G uv, uv is never a selfish vertex for each $v \in V(G)$ such that $u \perp v$.

III. RESULTS

In this section we show that every graph is an induced subgraph of a DDS graph, and we discuss few methods of generating new DDS graphs from existing DDS graphs.

Theorem 1

Every graph is an induced subgraph of a DDS graph.

Proof

Let G be a graph with n vertices say u_i, where i = 1, 2, ..., n. Consider n copies of P_2. Label the vertices in P_2 as $v_i\, w_i\, z_i$, where i = 1, 2, …, n. Obtain a new graph H by merging $u_i v_i$ for all i = 1, 2, ..., n. Label the vertices $u_i v_i$ as x_i, i = 1, 2, ..., n. Note that $\{ w_i$, i = 1, 2, …, n$\}$ is the only γ - set for H. Hence $\gamma(H) = n$.

Consider H $z_i w_i$.$\gamma(H) = \gamma(H\ z_i w_i)$ i = 1, 2, … , n, since $\{ w_1, w_2, \ldots, w_{i-1}, w_i z_i, w_{i+1}, \ldots, w_n \}$ is a γ - set for H $z_i w_i$.

Consider H $x_i w_i$.$\gamma(H) = \gamma(H\ x_i w_i)$, i = 1, 2, … , n, since $\{ w_1, w_2, \ldots, w_{i-1}, x_i w_i, w_{i+1}, \ldots, w_n \}$ is a γ - set for H $x_i w_i$.

Consider H $x_i x_j$. $\gamma(H) = \gamma(H\ x_i x_j)$. $i \neq j$, i , j = 1, 2, …, n, since $\{ w_i \mid i = 1,2 \ldots, n \}$ is a γ - set for H $x_i x_j$.

Thus $\gamma(H) = \gamma(H\ ab)$ for all a , $b \in H$, $a \perp b$. Hence H is dominating dot stable and every graph is an induced subgraph of a DDS graph. ■

Theorem 2

Let G be a DDS graph. The graph H obtained by attaching P_1 to a good vertex of G is DDS.

Proof

Let G be DDS and let $D = \{ u_1, u_2, \ldots, u_k \}$ be a γ - set for G. Obtain a new graph H by attaching a pendant vertex v to a vertex in D say u_i. Now $\gamma(H\ u_i v) = \gamma(H)$ as u_i dominates v. $\gamma(H\ xy) = \gamma(H)$, $x, y \in V(G)$, $x \perp y$. [As we know by [R2], there is a γ - set including u_i or $u_i y$ which will dominate vertex v]. Hence H is DDS. ■

Remark

If u is a bad vertex and we attach a pendant vertex v at u then the resulting graph H is not DDS. Since $\gamma(H) = \gamma(G) + 1$ but $\gamma(H\ uv) = \gamma(G)$.

Example

In Fig. 2. $\gamma(G) = 1$, $\gamma(H) = 2$, that is H uv = G and $\gamma(H\ uv) = 1$.

Theorem 3

Let G be a DDS graph. The graph H obtained by attaching a path P_2 to a bad vertex in G is DDS.

Proof

Let G be DDS and x be a bad vertex of G. Attach a path P_2: xvw at x to obtain a new graph H. Note that x is not a down vertex. Clearly $\gamma(H) = \gamma(G) + 1$. Consider H xv. We have $D \cup \{ w \}$ is γ - set for H yv, where D is a γ - set of G. Hence $\gamma(H\ xv) = \gamma(H)$. Note that $\gamma(H\ xv) = \gamma(H\ vw) = \gamma(H)$ .We know that G is DDS, also x is a bad vertex. By [R3], in G xy, the vertex xy is never selfish for all $y \perp x$ in G. So $\gamma(H\ xy) = \gamma(H)$, for all $y \in V(G)$ such that $x \perp y$. As G is DDS, for all $a, b \in V(G) - \{ x \}$, $a \perp b$, $\gamma(H\ ab) = \gamma(H) = \gamma(G) + 1$. Thus H is also DDS graph. ■

Example

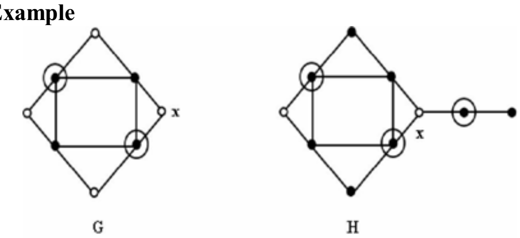


Fig. 3. x is a bad vertex, $\gamma(G) = 2$, $\gamma(H) = 3$ and $\gamma(H) = \gamma(H\ ab) = 3$ for all $a, b \in V(H)$, $a \perp b$.

Remark

Let G be a DDS graph and x be a good vertex. Then the graph H obtained by attaching a path P_2 to the vertex x is not DDS.

Proof

Let G be a DDS graph and x be a good vertex. Attach a path P_2 : xvw at x of G to obtain a new graph H. There is a γ - set D of G containing x. Hence $\gamma(H) = \gamma(G) + 1$, since $D \cup \{ v \}$ is a γ - set for H. $\gamma(H\ uw) = \gamma(H) - 1$, since D is also a γ - set of H uw. Hence $\gamma(H\ uw)$ $\gamma(H)$. Hence H is not DDS. ■

Theorem 4

Let G be a DDS graph. Then the graph H obtained by attaching a path P_3 to any vertex u in G is DDS.

Proof

Let G be a DDS graph with n vertices. Let D be a γ – set for G. Attach a path P_3 : uwxy to any arbitrary vertex u in G. Then we get a new graph H. Note that $\gamma(H\ uw) = \gamma(H\ wx) = \gamma(H\ xy) = \gamma(H) = \gamma(G) + 1$, as x is not a down vertex of G. Let $a, b \in V(G)$, $a \perp b$. Then $\gamma(G) = \gamma(G\ ab)$. Let D be γ - set for G ab. $D \cup \{ x \}$ is a γ - set for H ab. So $\gamma(H\ ab) = \gamma(G\ ab) + 1 = \gamma(G) + 1 = \gamma(H)$. Hence H is DDS. ■

Theorem 5

Let G be a DDS graph. Consider a path P_4: $w_1\, w_2\, w_3\, w_4\, w_5$ of length four with $V(G) \cap V(P) =$. The graph H obtained by joining a vertex $u \in V(G)$ and w_3 by an edge is DDS.

Proof

Let G be a DDS graph. Let D be a γ - set of G. Consider a path $P_4 : w_1\ w_2\ w_3\ w_4\ w_5$. Let $u \in V(G)$ and let H be the graph obtained by joining u and w_3 by an edge e. $\gamma(H) = \gamma(G) \cup \{w_2\} \cup \{w_4\}$ which implies $\gamma(H) = \gamma(G) + 2$.

Consider $H\ \ w_1w_2$. We have $\gamma(G) \cup \{w_1w_2\} \cup \{w_4\}$ is a γ - set for $H\ \ w_1w_2$ which implies $\gamma(H\ \ w_1w_2) = \gamma(H)$.

Consider $H\ \ w_2w_3$. We have $\gamma(G) \cup \{w_2w_3\} \cup \{w_5\}$ is a γ-set for $H\ \ w_2w_3$ which implies $\gamma(H\ \ w_2w_3) = \gamma(H)$.

Consider $H\ \ w_3w_4$. We have $\gamma(G) \cup \{w_3w_4\} \cup \{w_1\}$ is a γ-set for $H\ \ w_3w_4$ which implies $\gamma(H\ \ w_3w_{24}) = \gamma(H)$.

Consider $H\ \ w_4w_5$. We have $\gamma(G) \cup \{w_4w_5\} \cup \{w_2\}$ is a γ - set for $H\ \ w_4w_5$ which implies $\gamma(H\ \ w_4w_5) = \gamma(H)$.

Consider $H\ \ w_3u$. We have $\gamma(G) \cup \{w_4\} \cup \{w_2\}$ is a γ - set for $H\ \ w_3u$ which implies $\gamma(H\ \ w_3u) = \gamma(H)$.

In general $\gamma(H\ \ ab) = \gamma(H)$ for all $a, b \in V(H)$, $a \perp b$, that is H is a DDS graph. ■

Example

Fig. 4. $\gamma(G) = 1$, $\gamma(H) = 3$ and $\gamma(H) = \gamma(H\ \ uv) = 3$, which implies H is DDS.

Theorem 6

Let G_1 and G_2 be DDS graphs and $u \in V(G_1)$ and $v \in V(G_2)$. Let $V(G_1) \cap V(G_2) = \phi$. Obtain a graph H from G_1 and G_2 by joining the vertex u of G_1 and the vertex v of G_2 by an edge. Then H is also DDS if and only if either u is a bad vertex of G_1 or v is bad vertex of G_2.

Proof

Let G_1 and G_2 be two DDS graphs with $V(G_1) \cap V(G_2) = \phi$. Let $u \in V(G_1)$ and $v \in V(G_2)$. Join u and v by an edge to obtain a new graph H. We claim that $\gamma(H) = \gamma(G_1) + \gamma(G_2)$. Clearly $\gamma(H) \leq \gamma(G_1) + \gamma(G_2)$. Let D be a γ - set for H. Let $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$.

If $u \in D_1$ and $v \in D_2$ (or if both $u \notin D_1$ and $v \notin D_2$), then D_i is a dominating set for G_i, i = 1, 2. So $|D| = |D_1| + |D_2| \geq \gamma(G_1) + \gamma(G_2)$.

If $u \notin D_1$, then D_1 is a dominating set of $G - u$ and D_2 is a dominating set for G_2. As G_1 is a DDS graph, u is not a down vertex in G_1, so $|D_1| \geq \gamma(G_1)$ and $|D_2| \geq \gamma(G_2)$. Therefore $|D| = |D_1| + |D_2| \geq \gamma(G_1) + \gamma(G_2)$. Thus $\gamma(H) \geq \gamma(G_1) + \gamma(G_2)$ and hence $\gamma(H) = \gamma(G_1) + \gamma(G_2)$.

Case 1

Assume that either u is a bad vertex in G_1 or v is a bad vertex in G_2.

We claim that H is also DDS. It is enough to prove this by assuming u is a bad vertex of G_1. If $x \perp y$ in $V(G_1)$, then $H\ \ xy$ is obtained by joining u of $G_1\ \ xy$ and v of G_2 by an edge and hence $\gamma(H\ \ xy) = \gamma(G_1\ \ xy) + \gamma(G_2) = \gamma(G_1) + \gamma(G_2) = \gamma(H)$, as G_1 is DDS. Similarly result holds if $x \perp y$ in $V(G_1)$. It is enough to prove that $\gamma(H\ \ uv) = \gamma(G_1) + \gamma(G_2) = \gamma(H)$. Let D be a γ - set for $H\ \ uv$.

If $uv \notin D$, then $D \cap V(G_i) = D_i$ is dominating set for G_i and $|D| = |D_1| + |D_2| \geq \gamma(G_1) + \gamma(G_2)$. So in this case $\gamma(H\ \ uv) = \gamma(H)$.

If $uv \in D$, let $D_1 = (D - uv) \cap V(G_1)$ and $D_2 = (D - uv) \cap V(G_2)$. Then $D_1 \cup \{u\}$ and $D_2 \cup \{v\}$ are dominating set of G_1 and G_2 respectively. As u is a bad vertex of G_1, $|D_1 \cup \{u\}| \geq \gamma(G_1) + 1$ and $|D_2 \cup \{v\}| \geq \gamma(G_2)$.

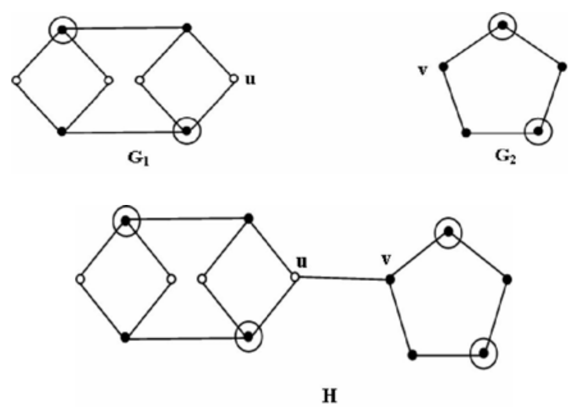
Therefore $|D| = |D_1| + |D_2| + 1 = |D_1 \cup \{u\}| + |D_2 \cup \{v\}| - 1 \geq \gamma(G_1) + 1 + \gamma(G_2) - 1 = \gamma(G_1) + \gamma(G_2)$.

Hence $\gamma(H\ \ uv) = \gamma(G_1) + \gamma(G_2) = \gamma(H)$. Thus H is DDS.

Case 2

Let u be a good vertex of G_1 and v be a good vertex of G_2. Let D_1 and D_2 be γ - sets of G_1 and G_2 containing u and v respectively. Then $(D_1 - \{u\}) \cup (D_2 - \{v\}) \cup \{uv\}$ is a dominating set for $H\ \ uv$. So $\gamma(H\ \ uv) \leq \gamma(G_1) + \gamma(G_2) - 1 < \gamma(H)$. So H is not DDS. ■

Example



In Fig. 5. $\gamma(G_1) = \gamma(G_2) = 2$, which implies $\gamma(H) = \gamma(G_1) + \gamma(G_2)$ that is $\gamma(H) = \gamma(H\ \ uv) = 4$.

Theorem 7

Let G_1 and G_2 be DDS graphs. Let $u \in V(G_1)$ and $v \in V(G_2)$. Let H be a graph generated by merging u and v. Then H is DDS if and only if u is a bad vertex of G_1 or v is a bad vertex of G_2.

Proof

The proof is similar to theorem 6. ■

IV. CONCLUSION

There are many possible variation of the idea. One can for instance, generate new dot stable graphs by attaching any tree that is dot stable, either by adding an edge or by vertex fusion. Just like the operations to build the larger graphs from smaller ones, we can also do the opposite, decompose a large graph that is dot stable to smaller graph that are not dot stable graph.

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Dr. M. Yamuna received her Ph.D. from Alagappa University, Karaikudi, India. Presently she is an Assistant Professor (Sr), VIT University, Vellore, India. K.Karthika received her M.Phil in Mathematics from VIT University, Vellore, India. Presently she is a Assistant Professor (Jr), VIT University. Both authors are interested in Graph Domination.