# Capacity of Gaussian Relay Channel

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***Abstract*-In commercial wireless communication system, most radios operate in time division duplex (TDD) mode that cannot transmit and receive at the same time in the same frequency band. We present a type of Gaussian degraded relay channel with nodes using light radios.The achievability of Gaussian degraded light relay channel is proved by using superposition encoding and successive decoding techniques. The converse is derived using the min-cut max-flow theorem for networks with light nodes. Our work of capacity analysis shows that even with light radios, our technique of coding is beneficial and has the advantage of capacity over direct transmission.**

## I. INTRODUCTION

The relay channel was first introduced by Van der Meulen [1] and investigated extensively by Cover and El Gamal [2]. Cover and El Gamal explored a number of relaying strategies, found achievable regions and provided upper bounds to the capacity of general relay channels. After that several researchers have focused on more complex types of relay channel [3], [4], and [5] but most of them have taken into account the use of full duplex radios *i.e*. which can send and receive at the same time in the same frequency band. Although it is possible to have full duplex relay network, but the design of such radios are not too favourable as they requireaccurate interference cancellation between transmitted and received signals that differ in power by several orders of magnitude. We consider a TTD technique of communication that cannot transmit and receive at the same time in the same frequency band.

To quantify the capacity of the network we consider an information theoretic approachhaving the constraint that the nodes of the network can either send or receive at a given time. We label such a radio as a *light* radio and the corresponding node of the network as a *light* node. For the sake of completeness and ease to understand, we follow the formulation and notation of [4], and address the capacity for discrete memoryless channels and extend the results to Gaussian channels.

First, we derive an upper bound on the capacity of the Gaussian light relay channel. Then, we derive thecapacity of the Gaussian degraded light relay channel.The bound, the proof of achievability derived in Theorem 2, bases on the coding scheme in [6]. Since, we do not use the assumption of degradeness in the proof of the coding theorem, so this rate can be interpreted as a lower bound to the capacity of the Gaussian light relay channel in general. By applying the assumption of degradeness, the converse is proved with slight changes from the proof of the derived upper bound on the capacity of Gaussian light relay channel. Our results show that even in light relay channels, there is a sizable gain in capacity if we use the relay.

The rest of this paper is organized as follows: after putting a bird eye view over system model in the Section II, we drive an upper bound on the capacity of the light relay channel in the Section III. In the Section IV we derive the capacity of the Gaussian degraded light relay channeland in section V we present the proof of achievability of the capacity for Gaussian light relay channel. Conclusion is presented in section VI.

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## II. SYSTEM MODEL

We consider a single relay Gaussian network shown in Fig. 1. In this simple relay channel, the source node S intends to transmit information to destination node D by using the direct link between the node pair (S, D) as well as the help of the relay node R (in the case if it improves the achievable rate of transmission) by using link pair (S, R) and (R, D).

We consider a communication system where the intermediate relay node operates in TDD mode when transmitting and sending in same frequency band. The light relay channel, shown in figure 1, consists of an input $x_1$, a relay output $y_2$, a relay sender $x_2$ and a channel output $y_3$. The channel is assumed to be memoryless. We model a half-duplex operation using the state variable $Q$ that controls the relay operation. $Q$ takes the value $q_1$ if the relay is listening and $q_2$ if the relay is transmitting. We also consider fixed protocols, in which the relay listens for a fixed time interval fraction $t$, ($0 \leq t \leq 1$) and then transmits in the remaining portion (1-$t$).The two states of operation are depicted in figure 2.

The dependencies of the outputs on the inputs are as follows: In $q_1$state, relay node R acts as a receiver and thus the channel output is given by$y_3 = h_1x_1 + z_3$ and the relay output is given by$y_2 = h_0x_1 + z_2$.In state $q_2$the relay node function as a transmitter and the channel output is given by$y_3 = h_1x_1 + h_2x_2 + z_3$ Where $h_0, h_1$ and $h_2$ are channel losses and are assumed to be constant, and $z_3 \sim N(0, N_3)$ and $z_2 \sim N(0, N_2)$are independent Gaussian noises. Suppose that in a total of$n$channel uses, the channel is used $k$times in state$q_1$, and $n - k$times in state $q_2$. We assume the following three power constraints

(i) $\frac{1}{k}\sum_{i=1}^{k} x_{1i}^2 | q_1 \leq P_0$

(ii) $\frac{1}{n-k}\sum_{i=1}^{n-k} x_{1i}^2 | q_2 \leq P_1$

(iii) $\frac{1}{n-k}\sum_{i=1}^{n-k} x_{2i}^2 | q_2 \leq P_2$

Where $x_{ji|q_K}$ is the $i^{th}$ transmitted signal from node j while channel is used in state $q_k$. The problem is to find out the channel capacity between sender S and receiver D.

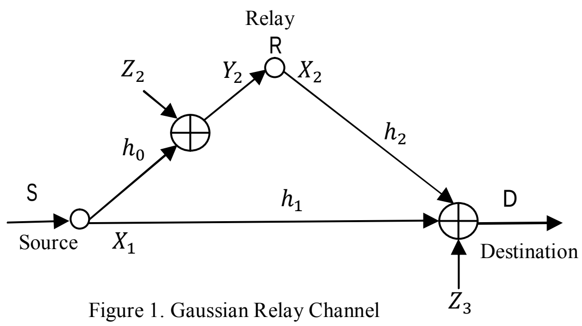


Figure 1. Gaussian Relay Channel

Let $q_1 = 0, q_2 = 1$, $\mathcal{S} = \{q_1, q_2\}$ and $\mathcal{R}$ be the set of real numbers. An $(M, n)$ code for a Gaussian memoryless light relay channel consists of a set of integers

$$\mathfrak{M} = \{1,2, \ldots\ldots M\} \triangleq [1, M]$$

an encoding function

$$s^n : \mathfrak{M} \longrightarrow \mathcal{S}^n \quad , \quad x_1^n : \mathfrak{M} \longrightarrow \mathcal{R}^n$$

Where $x_1^n$ denotes a n-tuple $(x_{11}, x_{12}, \ldots\ldots x_{1n})$ and $s_i$ is the state of the network at the $i^{th}$ network use, a set of relay function $\{f_i\}_{i=1}^n$ such that

$$x_{2i} = f_i(Y_{21}, Y_{22}, \ldots . Y_{2(i-1)}) \ for\ 1 \leq i \leq n$$

Where $y_{2i}$ is the received signal at the relay and $x_{2i}$ is the transmitted signal from the relay at time $i$

andthe decoding function

$$g : (y^n, \mathcal{S}^n) \longrightarrow \mathfrak{M}$$

For generality, the encoding functions $x_i(.), f_i(.)$and decoding function $g(.,.)$ are allowed to be stochastic functions.

Note that the structure of both the encoding functions at the source and the relay nodes are actually from the definition of the light relay channel. At the source node, encoding is based on choosing the mode of operation of the network and then choosing the symbols in the corresponding mode. Also, because of non-anticipatory relay condition, relay input $x_{2i}$is allowed to depend only on the past $y_2^{(i-1)} = (y_{21}, y_{22}, \ldots . y_{2(i-1)})$values of the received signals [2].

## III. Capacity upper bound

According to [4] an upper bound for the information transfer rate R

$$R \leq \max_{t, 0\leq t\leq 1} \min \{tI(X_1; Y_3, Y_2 | q_1) + (1-t)\, I(X_1; Y_3 | X_2, q_2),$$
$$tI(X_1; Y_3 | q_1) + (1-t) I(X_1, X_2; Y_3 | q_2)\} \quad (1)$$

is achievable for a fixed input distribution and a fixed *t*, can be maximized over all input distributions, $t \in [0,1]$. The first term in (1) is the sum of two mutual information expressions. The first one indicates the amount of information the relay can decode in *t* fraction of the time. The second is the mutual information the destination collects from the source during the time when the relay is transmitting. The second term in (1) is also a sum of two terms, the first of which is the mutual information at the destination while the relay is silent, and the second is the rate at which the source and the relay can together send in 1-*t* fraction of time. Similar to full-duplex case, if the half-duplex relay channel is physically degraded, then the above rate is capacity achieving.

To express the bound in term of power constraints in a Gaussian light relay channel, we use the following bounds from [2, eq.(53)] at the rate R

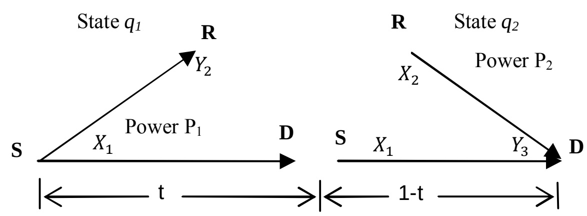


Figure 2. 2-modes of Light Relay channel

$$n\,R \leq \max_{k, 0\leq k\leq n} min \left\{ \sum_{i=1}^{k} I(X_1; Y_3, Y_2 | q_1) + \sum_{i=1}^{n-k} I(X_1; Y_3 | X_2, q_2), \right.$$
$$\left. \sum_{i=1}^{k} I(X_1; Y_3 | q_1) + \sum_{i=1}^{n-k} I(X_1, X_2; Y_3 | q_2) \right\} + n\epsilon_n \qquad (2)$$

In term of power constraints and the channel parameters, the upper bound for the mutual information term $I(X_{1i}; Y_{3i}, Y_{2i} | q_1)$for $1 \leq i \leq n$can be determined as [2]

$$\sum_{i=1}^{k} I(X_{1i}; Y_{3i}, Y_{2i} | q_1)$$

$$\sum_{i=1}^{k} I(X_{1i}; Y_{3i}, Y_{2i} | q_1)$$
$$\leq \frac{k}{2} log\left(1 + \frac{|h_0|^2 P_0}{N_2} + \frac{|h_1|^2 P_1}{N_3}\right) \qquad (3)$$

Similarly in term of power constraints and the channel parameters, the upper bound for the mutual information term $I(X_{1i};Y_{3i}|X_{2i},q_2)$for $1 \leq i \leq n$can be find out as [2]

$$\sum_{i=1}^{n-k} I(X_{1i};Y_{3i}|X_{2i},q_2)$$

$$\leq \frac{n-k}{2} log\left(1 \frac{(1-\rho^2)|h_1|^2 P_1}{N_3}\right) \quad (4)$$

The upper bound for the mutual information term $I(X_{1\,i};Y_{3\,i}|q_1)$for $1 \leq i \leq n$ can be find out as [2]

$$\sum_{i=1}^{k} I(X_{1i};Y_{3i}|q_1) \leq \sum_{i=1}^{k}[h(Y_{3i}) - h(Y_{3i}|X_{1i})]$$

$$\sum_{i=1}^{k} I(X_{1i};Y_{3i}|q_1) \leq \frac{k}{2} log\left(1 + \frac{|h_1|^2 P_0}{N_3}\right) \quad (5)$$

and the upper bound for the mutual information term $I(X_{1i},X_{2i};Y_{3i}|q_2)$for $1 \leq i \leq n$ can be find out as [2]

$$\sum_{i=1}^{n-k} I(X_{1i},X_{2i};Y_{3i}|q_2) \leq \sum_{i=1}^{n-k}[h(Y_{3i}) - h(Y_{3i}|X_{1i},X_{2i})]$$

$$\leq \frac{n-k}{2} log\left(1 + \left(\frac{|h_1|^2 P_1}{N_3} + \frac{|h_2|^2 P_2}{N_3} + \frac{2|h_1 h_2|}{N_3}\frac{1}{n-k}\sum_{i=1}^{n-k}\sqrt{E[X_{2\,i}^2]E[E^2[X_{1\,i}/X_{2\,i}]]}\right)\right)$$

$$\leq \frac{n-k}{2} log\left(1 + \left(\frac{|h_1|^2 P_1}{N_3} + \frac{|h_2|^2 P_2}{N_3} + \frac{2|h_1 h_2|\rho\sqrt{P_1 P_2}}{N_3}\right)\right) \quad (6)$$

Now by substituting the calculated bounds for the mutual from (3),(4),(5)and(6)into (2),we have

$$n\,R \leq \max_{k,0\leq k\leq n} min\left\{\frac{k}{2} log\left(1 + \frac{|h_0|^2 P_0}{N_2} + \frac{|h_1|^2 P_0}{N_3}\right) + \frac{n-k}{2} log\left(1 + \frac{(1-\rho^2)|h_1|^2 P_1}{N_3}\right), \frac{k}{2} log\left(1 + \frac{|h_1|^2 P_0}{N_3}\right) + \frac{n-k}{2} log \left(1 + \left(\frac{|h_1|^2 P_1}{N_3} + \frac{|h_2|^2 P_2}{N_3} + \frac{2|h_1 h_2|\rho\sqrt{P_1 P_2}}{N_3}\right)\right)\right\} + n\epsilon_n \quad (7)$$

Diving both sides of bound in (7) by $n$ and let$n \longrightarrow \infty$ , yields the following theorem

**Theorem 1**: An upper bound for the capacity of Gaussian light relay channel of fig 1 with input-output relations and power constraints defined in section II is given by

$$R \leq \frac{1}{2}\max_{t,\rho,0\leq t,\rho\leq 1} min\{t log(1 + (\gamma_0 + \gamma_1) P_0) + (1-t) log(1 + (1-\rho^2)\gamma_1 P_1), t log(1 + \gamma_1 P_0) + (1-t) log(1 + \gamma_1 P_1 + \gamma_2 P_2 + 2\rho\sqrt{\gamma_1\gamma_2 P_1 P_2})\} \quad (8)$$

Where $\gamma_0 \triangleq \frac{|h_0|^2}{N_2}$, $\gamma_1 \triangleq \frac{|h_1|^2}{N_3}$ ,$\gamma_2 \triangleq \frac{|h_2|^2}{N_2}$

IV.Gaussian Degraded light relay channel

Now suppose that the channel transmitter at node S with some power constraints wants to send a signal to ultimate receiver D, but the relay receiver R, which may be physicallycloser to node S, also receive the same signal. In such a scenario of information transmission, we say that the received signal at node D is degraded form of signal received at node relay R. We derive the capacity of this type of Gaussian degraded light relaychannel as:

**Theorem 2:** The capacity of Gaussian degraded light relay channel (fig 3) with input- output relations and power constraints defined in section II, can be derived in (10) with achievable rate as follows:

$$R \leq \max_{t,\rho,0\leq t,\rho\leq 1} min\left\{t\mathrm{C}(P_{SR_1})(1-t)C(1-\rho^2)P_{SD_2}, tC(P_{SD_1}) + (1-t)C\left(P_{SD_2} + P_{R_2D_2} + 2\rho\sqrt{P_{SD_2} + P_{R_2D_2}}\right)\right\} \quad (9)$$

Where $\rho$ the correlation between the source and relay signals in state $q_2$. We assume that the source node S transmit with power $P_0$ and $P_1$ during the state $q_1$ and $q_2$respectively, and the relay node transmits with power $P_2$. The received power notations are:

$$P_{SR_1} = \gamma_0 P_0, P_{SD_2} = \gamma_1 P_1, P_{SD_1} = \gamma_1 P_0, \quad P_{R_2D_2} = \gamma_2 P_2$$

Now

$$C = \frac{1}{2}\max_{t,\rho,0\leq t,\rho\leq 1} \min\left\{\{t log((1 + \gamma_0 P_0)(1-t) log (1 + (1-\rho^2)\gamma_1 P_1), t log(1 + \gamma_1 P_0) + (1-t) log (1 + \gamma_1 P_1 + \gamma_2 P_2 + 2\rho\sqrt{\gamma_1\gamma_2 P_1 P_2})\}\right. \quad (10)$$

Where $C(x) = \frac{1}{2}\log(1+x)$is the capacity of a Gaussian link.

In next section we prove the achievability of any rate $R \leq C$ in (10) whether the channel is degraded or not. Thus this achievable rate is considered as the lower bound to the capacity of Gaussian light relay channel, and with the assumption of degraded relay channel the upperbound of theorem 1 can be reduced as $I(X_{1i}; Y_{3i}, Y_{2i}|q_1) = I(X_{1i}; Y_{2i}|q_1)$ and with the analysis similar made in section III-C, the bound of theorem 1 reduces to the expression of C in (10).

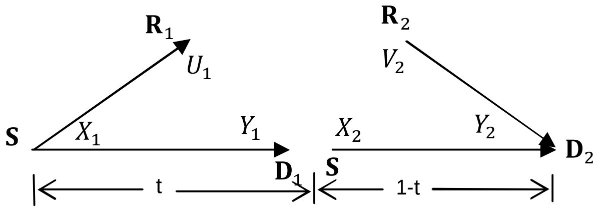


Figure 3. Two modes of operation for light Relay Channel

V. Achievability of C in theorem 2

Here we show the achievability for the rate R$\leq$ C in theorem 2 for any $t$, $0 \leq t \leq 1$ .The proof is based on a coding scheme similar to the coding scheme in [6].

**Theorem 3:**The capacity of the degraded light relay channel is given by [6].

$$C = \max_{t,p(.)} min\{t\ I(X_1; U_1|q_1) + (1-t)I(X_2; Y_2|V_2, q_2),$$

$$t\ I(X_1; Y_1|q_1) + (1-t)\,I(X_2, V_2; Y_2|q_2) \quad (11)$$

Where supremum is taken over $t$, $0 \leq t \leq 1$ and the channel probabilities are $p(x_1, u_1, y_1) = p(x_1)p(u_1, y_1|x_1)$ and $p(x_2, v_2, y_2) = p(x_2, v_2)p(y_2|x_2, v_2)$

The coding technique that achieves (11) is as that first we split the source information into two independent parts, say$(w, v)$. A total of $N$ symbols are transmitted, $tN$symbols (we suppose that$tN$ is an integer)are transmitted in $q_1$state with a rate $R_{SR_1}$while the remaining part of information is sent in $q_2$state with a rate $R_{SD_2}$. The average rate in $q_1$ and $q_2$ states equal to $R_{SR_1} + R_{SD_2}$ bit per channel use.For sake of simplicity we shall confine to the rates of component code in mutual information terms.

V. ENCODING AND DECODING IN Q1STATE

In $q_1$state, the source encodes $w$to produce a $tN$symbol length codeword$CSR_1 \in \mathcal{C}_{SR_1}$ with rate

$$R_{SR_1} = I(X_1; U_1|q_1)(12)$$

The codeword$CSR_1$ with added noise is received by both the relay and the destination. Since $R_{SR_1}$ is an achievable rate for the SR link, the relay can decode $CSR_1$ reliably. However, the destination cannot decode it because of its poor capacity. The destination, therefore, stores the received signal for decoding at the end of $q_2$ state,

IV. ENCODING AND DECODING IN Q2 STATE

The destination already has $tNI(X_1; Y_1)$bits of information from $q_1$ state in (16). So it needs an additional $tN\big(I(X_1; U_1) - I(X_1; Y_1)\big)$bits to reliably decode $CSR_1$. These extra bits are sent jointly by the source and the relay using a codeword$CRD_2 \in \mathcal{C}_{RD_2}$ which is sent jointly by nodes S and R to node D at a rate of

$$R_{RD_2} = I(X_1; U_1) - I(X_1; Y_1) \quad (13)$$

or

$$R_{RD_2} = I(V_2; Y_2|X_2, q_2) \quad (14)$$

The second part of the information$v$ is also sent in $q_2$state using a codeword$CSD_2 \in \mathcal{C}_{SD_2}$ which utilizes the remaining capacity of the channel that consists of the source and relay as transmitters and the destination as the receiver. The new information$v$ is sent by source alone. The rate of this information is

$$R_{SD_2} = min\{I(X_2, V_2; Y_2|q_2)I(V_2; Y_2|X_2, q_2), I(X_2; Y_2|V_2, q_2)\} \quad (15)$$

(12) and (15) gives the average rate in two states of channel operation. The destination first decodes the codewords$CRD_2$ and $CSD_2$ transmitted in $q_2$state. The source and relay rates in $q_2$state, $R_{SD_2}$ and $R_{RD_2}$ correspond to a point that can be achieved by successive decoding. After decoding $CSD_2$ and $CRD_2$ ,the destination can decode the noisy codeword$CSR_1$ using the information carried by $CRD_2$ as side information.For decoding $CSR_1$ ,the destination treats it as a codeword $CSD_1 \in \mathcal{C}_{SD_1}$of rate

$$R_{SD_1} = I(X_1; Y_1/q_1) \quad (16)$$

Now for some $\rho$, $0 \leq \rho \leq 1$Let $X_W \sim N(0, (1-\rho^2)P_0)$, and $X_V \sim N(0, P_2)$ with $W$ and $V$ independentpair of source messages.

Let

$X_1/q_1 = \sqrt{P_0/P_2 X_V}$ and

$X_1/q_2 = \sqrt{\rho^2 P_1/P_2 X_V} + X_W$ and $X_2/q_2 = X_V$

Referring to theorem 3, we evaluate

$I(X_1; U_1|q_1) = \dfrac{1}{2}\log(1 + \gamma_0 P_0)$

$$I(X_2; Y_2|V_2, q_2) = \frac{1}{2}\log(1 + (1 - \rho^2)\gamma_1 P_1)$$

$$I(X_1; Y_1|q_1) = \frac{1}{2}\log(1 + \gamma_1 P_0)$$

$$I(X_2, V_2; Y_2|q_2) = \frac{1}{2}log\left(1 + \gamma_1 P_1 + \gamma_2 P_2 + 2\rho\sqrt{\gamma_1\gamma_2 P_1 P_2}\right)$$

The assertion that this distribution for $X_1/q_1$ , $X_1/q_2$ and $X_2/q_2$ actually achieve the capacity C in (11) follows from the proof of the converse.

The random codebook associated with this distribution is then given by the random choice of

$$X_W(w)\ i.i.d. \sim N_n(0, (1 - \rho^2)P_0 I) \qquad w \in \left[1, 2^{n\,H(W)}\right]$$

$$X_V(v)\ i.i.d. \sim N_n(0, P_2 I) \qquad v \in \left[1, 2^{n\,H(V)}\right]$$

Where $N_n(0, C_n)$ denote the n-variant normal distribution with zero mean and covariance matrix $C_n$ .The codebook is given by

$$X_1/q_1(v) = \sqrt{\frac{P_0}{P_2}X_v(v)}\ ,\ X_1/q_2(u/v) = \sqrt{\frac{\rho^2 P_1}{P_2}X_v(v)} + X_U(u)$$

and $X_2/q_2(v) = X_V(v)$

The codewords so generated $(1-\in)$ satisfy the power constraints with higher probability and thus the overall average probability of error can be shown to be small.

## VI. CONCLUSION

In theorem 2, the capacity of the Gaussian degraded light relay channel is derived which is apparently less than the capacity of the conventional full duplex relay channel, but the lower bound on this capacity shows that it can provide higher capacity with respect to the direct link.As it is crystal clear from the derived capacity expressions that if $\gamma_1 > \gamma_0$ then the direct transmission, without using the Markovian scheme,shall achieve a higher rate. The achievable rate of direct transmission for the Gaussian relay channel is given by $R_{Direct} = \frac{1}{2}log(1 + \gamma_1 P_1)$ which implies that if relay is in a good situation in term of received SNR with respect to the destination received SNR *i.e.* $\gamma_0 > \gamma_1$, then using the relay is helpful and improves the achievable rate of the direct transmission. Thus, we may conclude that even though the relay nodes in the network are light, we can achieve a higher transmission rate by exploiting network coding over the transmitter and the relay nodes.

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